PERIODS OF ITERATED INTEGRALS OF HOLOMORPHIC FORMS ON A COMPACT RIEMANN SURFACE

BY

SHU-YIN HWANG-MA

ABSTRACT. Holomorphic forms are integrated iteratedly along paths in a compact Riemann surface M of genus g, thus inducing a homomorphism from the fundamental group $\Gamma = \pi_1(M, P_0)$ to a proper multiplicative subgroup G of the group of units in $\widehat{T(\Omega^{1^*})}$, where Ω^1 denotes the space of holomorphic forms on M, Ω^{1^*} is the complex dual of Ω^1, T means the associated tensor algebra and "$\widehat{}$" means completion with respect to the natural grading. The associated homomorphisms from $\Gamma/\Gamma^{(n+1)}$ to $G/G^{(n+1)}$ reduces to the classical case $H_1(M) \to \Omega^{1^*}$ when $n = 1$. We show that the images of $\Gamma/\Gamma^{(n+1)}$ are always cocompact in $G/G^{(n+1)}$ and are discrete for all $n > 2$ if and only if the Jacobian variety $J(M)$ of M is isogenous to E^g for some elliptic curve E with complex multiplication.

The point of view taken here is suggested by Parsin's paper [4], although that paper does not contain any result beyond the construction of a homomorphism. Also the cocompactness of the image of $\Gamma/\Gamma^{(n+1)}$ in $G/G^{(n+1)}$ follows from general results of Sullivan [8] and Chen on a similar setting for "real" differentiable manifolds and "real" nilpotent groups; however we give a different proof.

1. Iterated integrals. All maps and forms are assumed to be complex analytic here; tensor products are taken over $\mathbf{C}$ and linear means complex linear.

Let $T = \widehat{T(\Omega^{1^*})}$ and T_0 be the group of units in T. We will write elements in T as formal infinite series $l_0 + l_1 + l_2 + \cdots$ with $l_j \in \otimes^j \Omega^{1^*}$; then T_0 consists of elements with leading coefficient 1.

DEFINITION. An element $l = \Sigma l_n \in T$ is called a Lie element if $l_n = 0$ and l_n = combination of $[X_{i_1}, \ldots, X_{i_n}]$'s where $\{X_1, \ldots, X_g\}$ is a base for Ω^{1^*} and

$$\left[X_{i_1}, X_{i_2} \right] = X_{i_1} \otimes X_{i_2} - X_{i_2} \otimes X_{i_1}, \qquad \left[X_{i_1}, \ldots, X_{i_n} \right] = \left[\left[X_{i_1}, \ldots, X_{i_{n-1}} \right], X_{i_n} \right].$$

Chen [1] proved that the formal power series

$$\log\left(1 + \sum_{n > 0} \sum \alpha(i_1, \ldots, i_n) X_{i_1} \otimes \cdots \otimes X_{i_n} \right)$$

is a Lie element if the coefficients $\alpha(i_1, \ldots, i_n)$ are given by successive integration along a path.

$G = \{ l \in T | \log(l)$ is a Lie element$\}$ is a subgroup of T_0. We note that the filtration $\{ G^{(n)} \}$ is induced by the pullback of the natural filtration on T under the

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log map. Setting $\mathcal{F}$ = the free Lie algebra generated by Ω^{1*}, $\hat{\mathcal{F}}$ is the set of all Lie elements in T. We see that $\hat{\mathcal{F}}_n/\hat{\mathcal{F}}_{n+1} \cong G^{(n)}/G^{(n+1)}$ and

$$\hat{\mathcal{F}}_n/\hat{\mathcal{F}}_{n+1} \cong \big[\big[\ldots [\Omega^{1*}, \Omega^{1*}], \Omega^{1*}\big], \ldots, \Omega^{1*}\big] \qquad (n \text{ times}).$$

($\{\hat{\mathcal{F}}_n\}$ is the natural grading of $\hat{\mathcal{F}}$.)

Let $\gamma \in \Gamma$ and associate with it a collection of n-linear functionals ϕ_n of holomorphic 1-forms as follows:

$$\phi_0(\gamma) = 1, \qquad \phi_1(\gamma)(\omega) = \int_\gamma \omega = \int \gamma^*\omega,$$

$$\phi_n(\gamma)(\omega_1, \ldots, \omega_n) = \int_{S_n} (\gamma^n)^*(\omega_1 \otimes \cdots \otimes \omega_n) = \int_\gamma \omega_1 \otimes \cdots \otimes \omega_n$$

where $n > 0$ and $S_n = \{(t_1, \ldots, t_n) \in I^n | 0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1\}$. Each $\phi_n(\gamma)$ can be regarded as an element of $\otimes^n (\Omega^{1*})$. Then we get a homomorphism ρ from Γ to T_0 induced by the ϕ_i's (Parsin [4]). As in Ree [7], one can show that $\rho(\gamma) \in G$ for all $\gamma \in \Gamma$. Thus we arrive at a group homomorphism $\rho: \Gamma \to G$ where $G/G^{(2)} = \Omega^{1*}$.

2. Main results. We will choose on M once and for all g pairs of crosscuts $A_1, B_1, \ldots, A_g, B_g$ affording a canonical dissection of M. Then $\Gamma = \pi_1(M, P_0) = \langle A_1, \ldots, A_g, B_1, \ldots, B_g | \Pi_{j=1}^g [A_j, B_j] = 1 \rangle$. And let $\omega_1, \ldots, \omega_g$ be the normalized base of Ω^1 w.r.t. $A_1, \ldots, A_g, B_1, \ldots, B_g$; $X_1, \ldots, X_g$ be the base for Ω^{1*} dual to $\omega_1, \ldots, \omega_g$. Z will denote the normalized period matrix (z_{ij}) where $z_{ij} = \int_{B_i} \omega_j$.

Define $\omega[i] = \omega_i$ and

$$\omega[i_1, \ldots, i_n] = \omega[i_1, \ldots, i_{n-1}] \otimes \omega[i_n] - \omega[i_2, \ldots, i_n] \otimes \omega[i_1].$$

By a direct computation (or Ree [7]), one sees

$$[X_{i_1}, \ldots, X_{i_n}](\omega_{j_1} \otimes \cdots \otimes \omega_{j_n}) = (X_{i_1} \otimes \cdots \otimes X_{i_n})(\omega[j_1, \ldots, j_n])$$

and

$$[X_{i_1}, \ldots, X_{i_n}](\omega[j_1, \ldots, j_n]) = n(X_{i_1} \otimes \cdots \otimes X_{i_n})(\omega[j_1, \ldots, j_n]).$$

So we can identify $G^{(n)}/G^{(n+1)}$ with $\langle \omega[j_1, \ldots, j_n] | (j_1, \ldots, j_n) \in I \rangle^*$ where $I = \{1, \ldots, g\}^n$.

We define $[\gamma_1, \ldots, \gamma_n] = [[\ldots [\gamma_1, \gamma_2], \gamma_3], \ldots, \gamma_n]$ for $\gamma_1, \ldots, \gamma_n \in \Gamma$ and observe that

$$\rho([A_{j_1}, \ldots, A_{j_n}]) \equiv [X_{j_1}, \ldots, X_{j_n}] \mod G^{(n+1)}.$$

Further $\rho([A_{j_1}, \ldots, A_{j_{n-1}}, B_{j_n}]) \equiv \Sigma_k z_{j_n k}[X_{j_1}, \ldots, X_{j_{n-1}}, X_k]$.

Let I' be a subset of I such that the monomials

$$[X_{i_1}, \ldots, X_{i_n}]_{(i_1, \ldots, i_n) \in I'} \tag{1}$$

form a (complex) basis of the vector space of homogeneous Lie elements of degree n; that is, the space $G^{(n)}/G^{(n+1)}$. Then since the imaginary part of the period matrix (z_{ij}) is invertible, the set

$$\Big[X_{i_1}, \ldots, \sum_k z_{i_n k} X_k\Big]_{(i_1, \ldots, i_n) \in I'} \tag{2}$$

is also a complex basis, and the sets (1) and (2), taken together, form a basis of the real vector space $G^{(n)}/G^{(n+1)}$.

PROPOSITION 2.1. $\rho(\Gamma^{(n)})$ *is cocompact in* $G^{(n)}/G^{(n+1)}$.

If $Z = (z_{ij})$ has entries in $Z(\sqrt{-\alpha})$ for some $\alpha \in \mathbf{N}$, then it is trivial that $\rho(\Gamma^{(n)})$ is discrete in

$$G^{(n)}/G^{(n+1)}. \tag{3}$$

But since the image of $\Gamma^{(n)}$ in (3) is always finitely generated, an argument using a common denominator of the generators allows the same conclusion whenever the entries of Z lie in $\mathbf{Q}(\sqrt{-\alpha})$. So we have

PROPOSITION 2.2. *If* $Z \in M_g(\mathbf{Q}(\sqrt{-\alpha}))$ *for some* $\alpha \in \mathbf{N}$, *then the image of* $\Gamma^{(n)}$ *is discrete in* $G^{(n)}/G^{(n+1)}$.

For the converse, we first show

PROPOSITION 2.3. *If the image of* $\Gamma^{(2)}$ *is discrete in* $G^{(2)}/G^{(3)}$, *then all* $z_{ij} \in \mathbf{Q}(\sqrt{-\alpha})$ *for some* $\alpha \in \mathbf{N}$.

PROOF. $G^{(2)}/G^{(3)} \overset{\log}{\cong} [\Omega^{1*}, \Omega^{1*}] \cong [\Omega^1, \Omega^1]^*$. By the proof of Proposition 2.1 and the discreteness of $\rho(\Gamma^{(2)})$ in $G^{(2)}/G^{(3)}$, every element in the image of $\Gamma^{(2)}$ must be a rational combination of

$$\rho([A_i, A_j]) \equiv \Delta_{ij} \quad \text{and} \quad \rho([A_i, B_j]) \equiv S_{ij} \bmod G^{(3)}, \qquad 1 \leqslant j < i \leqslant g.$$

Write $\rho([A_1, B_k]) \equiv \sum_{i>j} \lambda_{ij}^k \Delta_{ij} + \sum_{i>j} \mu_{ij}^k S_{ij}$ where $1 \leqslant k \leqslant g$, $\lambda_{ij}^k, \mu_{ij}^k \in \mathbf{Q}$.

Taking the value at $[\omega_1, \omega_n]$, $1 < n \leqslant g$:

$$z_{kn} = -\lambda_{n1}^k - \sum_{j<n} \mu_{nj}^k z_{j1};$$

$$z_{12} = -\lambda_{21}^1 - \mu_{21}^1 z_{11}, \qquad (k, n) = (1, 2);$$

$$z_{1n} = -\lambda_{n1}^1 - \sum_{j<n} \mu_{nj}^1 z_{j1}, \qquad (k, n) = (1, n); \tag{2.3.1}$$

inductively, we see z_{1n} is a rational combination of 1 and z_{11} for $n > 1$; (2.3.1) then shows z_{kn} is a rational combination of 1 and z_{11}. Therefore, all $z_{ij} \in \mathbf{Q} + \mathbf{Q}z_{11}$. $Z = A + Bz_{11}$ with A and B in $M_g(\mathbf{Q})$.

Write $z_{11} = x_{11} + \sqrt{-1}\, y_{11}$ with $x_{11}, y_{11} \in \mathbf{R}$ and $y_{11} > 0$;

$$\rho([B_1, B_k]) \equiv \sum_{i>j} (c_{ij} \Delta_{ij} + d_{ij} S_{ij}), \qquad c_{ij}, d_{ij} \in \mathbf{Q};$$

taking value at $[\omega_m, \omega_n]$:

$$z_{1m} z_{kn} - z_{km} z_{1n} = -c_{nm} + \sum_{j<m} d_{mj} z_{jn} - \sum_{j<n} d_{nj} z_{jm}$$

is a rational combination of 1 and z_{11}.

Let $(m, n) = (1, 2)$, $(1, k) = (1, 2)$; we have

$$z_{11}(a_{22} + b_{22} z_{11}) - (a_{12} + b_{12} z_{11})^2 = a + b z_{11}.$$

where a_{ij}, b_{ij}, a, $b \in \mathbf{Q}$. So

$$a + bz_{11} = -a_{12}^2 + (a_{22} - 2a_{12}b_{12})z_{11} + \left(b_{22} - b_{12}^2\right)z_{11}^2,$$

$$b_{22} - b_{12}^2 = \frac{1}{y_{11}^2}\det\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} > 0.$$

Thus z_{11} satisfies a quadratic equation over $\mathbf{Q}$, $z_{11} = x + y\sqrt{-\alpha}$ for some x and y in $\mathbf{Q}$, $\alpha \in \mathbf{N}$ and $y > 0$. And hence $z_{ij} \in \mathbf{Q}(\sqrt{-\alpha})$ for all $i, j = 1, 2, \ldots, g$. Q.E.D.

PROPOSITION 2.4. $\rho(\Gamma)$ *is a lattice in* $G/G^{(n+1)}$ *for all* $n \geqslant 2$ *if and only if the image of* $\Gamma^{(n)}$ *is a lattice in* $G^{(n)}/G^{(n+1)}$ *for all* $n \geqslant 2$.

PROOF. If $\rho(\Gamma^{(n)})$ is a lattice in $G^{(n)}/G^{(n+1)}$ for all $n \geqslant 2$, then the cocompactness of the image of Γ in $G/G^{(n+1)}$ follows by induction from the fact that

$$G/(G^{(n+1)} \cdot \Gamma) \to G/(G^{(n)} \cdot \Gamma)$$

is a fibration with compact fibre.

To show the discreteness of $\rho(\Gamma)$ in $G/G^{(n+1)}$ we define H to be $G^{(n)}/G^{(n+1)} \cap$ Image of Γ and note that H is closed. We first show that H is discrete in $G^{(n)}/G^{(n+1)}$, hence in $G/G^{(n+1)}$.

$G^{(n)}/G^{(n+1)}$ is a vector space over $\mathbf{C}$ and $\rho(\Gamma^{(n)})$ is discrete and cocompact in $G^{(n)}/G^{(n+1)}$, $H \supset \rho(\Gamma^{(n)})$ contains $2 \times \dim_{\mathbf{C}}G^{(n)}/G^{(n+1)}$ linearly independent elements over $\mathbf{R}$, say, $a_1, \ldots, a_s$. It suffices to show every element in H is a rational combination of $a_1, \ldots, a_s$.

Assuming the negation, we can find an element $a \in H$, $a = \sum_{j=1}^{t}\lambda_j a_{i_j}$ where $i_1, \ldots, i_t \in \{1, \ldots, s\}$ and distinct $\lambda_1, \ldots, \lambda_t \in \mathbf{R}\setminus\mathbf{Q}$. Then, for any $c \in [0, 1]$, there are sequences of integers n_j, m_j^k, $j = 1, 2, \ldots$, $k = 1, \ldots, t$, such that

$$n_j a + \sum_{k=1}^{t} m_j^k a_{i_k} \to ca_{i_1} + a' \quad \text{as } j \to \infty$$

where a' is a real combination of $a_{i_2}, \ldots, a_{i_t}$. Because H is closed, $ca_{i_1} + a' \in H$ and H contains uncountably many elements. But $H \subseteq \rho(\Gamma/\Gamma^{(n+1)})$ where $\Gamma/\Gamma^{(n+1)}$ is a finitely generated nilpotent group, hence countable. We conclude that H is discrete.

Now to show $\rho(\Gamma)$ is discrete in $G/G^{(n+1)}$, we let $b_0 \in \rho(\Gamma/\Gamma^{(n+1)})$ and image of b_0 in $\rho(\Gamma/\Gamma^{(n)})$ be c_0. There is an open neighborhood V of c_0 in $G/G^{(n)}$ such that $V \cap \rho(\Gamma/\Gamma^{(n)}) = \{c_0\}$. If $b \in \rho(\Gamma/\Gamma^{(n+1)}) \cap$ preimage of V in $G/G^{(n+1)}$, then $b^{-1}b_0 \in H$. Since H is discrete, there is an open neighborhood U of the identity e in $G/G^{(n+1)}$ such that $U \cap \rho(\Gamma^{(n)}/\Gamma^{(n+1)}) = \{e\}$. $b_0U \cap$ preimage of V is then an open neighborhood of b_0 in $G/G^{(n+1)}$ and contains no other elements of $\rho(\Gamma/\Gamma^{(n+1)})$. Therefore $\rho(\Gamma/\Gamma^{(n+1)})$ is discrete.

Conversely, assume $\rho(\Gamma)$ is a lattice in $G/G^{(n+1)}$ for all $n \geqslant 2$; we have to show $\rho(\Gamma^{(n)})$ is a lattice in $G^{(n)}/G^{(n+1)}$ for all $n \geqslant 2$. In view of Proposition 2.1 we only have to show the discreteness, but this is obvious. Q.E.D.

So we have proved the main theorem:

THEOREM 2.5. *The image of* Γ *under* ρ *is a lattice in* $G/G^{(n+1)}$ *for all* $n \geqslant 2$ *if and only if* $Z \in M_g(K)$ *for some imaginary quadratic field* K.

The condition in the prior theorem can be interpreted as a condition on the Jacobian $J(M)$ by the following

PROPOSITION 2.6. $Z \in M_g(K)$ *for some imaginary quadratic field* K *if and only if* $J(M)$ *is isogenous to* E^g *where* E *is an elliptic curve with complex multiplication by an order in* K.

The proof of this proposition is nothing more than an easy checking of the definitions.

3. Examples. Fist we remark that the condition in Theorem 2.5 is neither null nor redundant. To be more precise, the Klein surface given by the equation $x^3y + y^3 + x = 0$ has a normalized period matrix

$$Z = \begin{bmatrix} -\dfrac{1}{8} + \dfrac{3\sqrt{7}}{8}\,i & -\dfrac{1}{4} - \dfrac{\sqrt{7}}{4}\,i & -\dfrac{3}{8} + \dfrac{\sqrt{7}}{8}\,i \\[2mm] -\dfrac{1}{4} - \dfrac{\sqrt{7}}{8}\,i & \dfrac{1}{2} + \dfrac{\sqrt{7}}{2}\,i & -\dfrac{1}{4} - \dfrac{\sqrt{7}}{4}\,i \\[2mm] -\dfrac{3}{8} + \dfrac{\sqrt{7}}{8}\,i & -\dfrac{1}{4} - \dfrac{\sqrt{7}}{4}\,i & \dfrac{7}{8} + \dfrac{3\sqrt{7}}{8}\,i \end{bmatrix}$$

(Rauch and Lewittes [6, pp. 297–308]) where one sees that all $z_{ij} \in \mathbf{Q}(\sqrt{-7}\,)$; while the Riemann surface of $y^2 = x^6 - x$ has a normalized period matrix

$$Z = \begin{bmatrix} \dfrac{1 - \sqrt{5} + i\sqrt{10 + 2\sqrt{5}}}{4} & \dfrac{2 - \sqrt{5} + i\sqrt{10 - 2\sqrt{5}}}{4} \\[3mm] \dfrac{3 - \sqrt{5} + i\sqrt{10 - 2\sqrt{5}}}{4} & \dfrac{i\sqrt{10 - 2\sqrt{5}}}{2} \end{bmatrix}$$

(Rauch [5, pp. 105–108, 155–159]) where $Z \notin M_2(K)$ for any imaginary quadratic field K.

By the results in Koblitz [3], we are able to list all the Fermat curves whose Jacobians are isogenous to E^g for some elliptic curve E with CM.

(3.1) The Jacobian J_d of the Fermat curve $x^d + y^d = 1$ is isogenous to E^g where $g = (d - 1)(d - 2)/2$ for some elliptic curve E with complex multiplication if and only if $d = 3, 4$ or 6.

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INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, NANKANG, TAIPEI, TAIWAN